ELSEVIER Journal of Computational and Applied Mathematics 82 (1997) 465–475

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

Stable spiral orbits of SOR Durand–Kerner's method applied to the equation $x^d = 0$

Y. Yamagishi *, [1]

Department of Applied Mathematics and Informatics, Ryukoku University, Seta, Ootu City, 520-21, Japan

Received 27 August 1996; received in revised form 15 March 1997

Abstract

The cardinality of stable spiral orbits for SOR Durand–Kerner's method is estimated for a small positive relaxation parameter. Experiments support that this estimation is still valid for Gauss–Seidel Durand–Kerner's method.

Keywords: Simultaneous polynomial root finding method; Durand–Kerner's method; Fixed points

AMS classification: 26C10; 30C15; 32H50; 39B12; 58F22

1. Introduction

Durand–Kerner's method is an iterative algorithm for finding zeros of a monic complex polynomial $p(x)$ of degree $d>1$. It was proposed independently by Weierstrass [7], Durand [2], Dochev [1], Kerner [4] and Prešić [6]. The definition of its original version is given in the appendix.

In this paper we consider Durand–Kerner's method with a modification called 'successive over relaxation'. This version of the method can be defined as the recursive formula

$$x_{i+d} = x_i - \lambda \frac{p(x_i)}{(x_i - x_{i+1}) \cdots (x_i - x_{i+d-1})}$$

that generates the sequence $\{x_i \in \mathbb{C} \mid i \geqslant 0\}$ of complex numbers with initial values $x_0, \ldots, x_{d-1}$. It can also be defined as the self-mapping of the complex affine space $\mathbb{C}^d$ to itself by

$$F(x_0, x_1, \ldots, x_{d-1}) = (x_1, x_2, \ldots, x_{d-1}, f),$$

* E-mail: yg@math.ryukoku.ac.jp.

[1] Partially supported by the Ryukoku University RIKOU Foundation.

PII S0377-0427(97)00088-5

where

$$f = x_0 - \lambda \frac{p(x_0)}{(x_0 - x_1)\cdots(x_0 - x_{d-1})}.$$

These two definitions have different viewpoints. We will use both of them, sometimes in a mixed manner.

The constant λ is called the relaxation parameter. If $\lambda = 1$, the method is especially called the Gauss–Seidel Durand–Kerner's method.

A geometrically modest property of this method was observed by Kanno et al. [3]: the trajectory $\{x_i\}$ behaves like a spiral as in Fig. 8 if the initial values lie on the vertices of a sufficiently large regular d-gon centered at the origin. The aim of this paper is to give it a mathematical description in the special case $p(x) = x^d$. (The spiral convergence may be compared to the "balanced convergence" of the original Durand–Kerner's method. See the appendix.)

In Section 2, we consider small $\lambda > 0$. Our result is that the number of "stable" spirals is equal to Euler's function $\varphi(d)$.

In Section 3 we give experiments. It is shown that if $\lambda = 1$ (Gauss–Seidel), the number of stable spirals still equals $\varphi(d)$. In addition, a "stable spiral of longer period" is found for $d \geqslant 4$.

2. Spiral orbits and their stability

Let $p(x) = x^d$. The mapping F has a property that the image under F of a line (i.e., one dimensional linear space) passing through the origin is again a line passing through the origin:

$$F(sx_0, sx_1, \ldots, sx_{d-1}) = sF(x_0, x_1, \ldots, x_{d-1}) \tag{1}$$

holds for any scalar $s \neq 0$ and any $(x_0, \ldots, x_{d-1})$ whose image under F is well-defined.

We take the variable change $y_i = x_i/x_{i-1}$, $i \geqslant 1$, with the "inverse" $x_i = x_0 y_1 \cdots y_i$, $i \geqslant 1$. (We are interested in the local behavior of dynamics of F in some subset of $\mathbb{C}^d$ where $x_0 \cdots x_{d-1} \neq 0$ holds.) By (1), the self-map of the y-space of "relative configurations" of dimension $d - 1$ is defined by

$$G(y_1, y_2, \ldots, y_{d-1}) = (y_2, \ldots, y_{d-1}, g),$$

where

$$g = \frac{1}{y_1 \cdots y_{d-1}} \left(1 - \frac{\lambda}{(1 - y_1)(1 - y_1 y_2)\cdots(1 - y_1 \cdots y_{d-1})}\right).$$

We can also say that the sequence $y_i \in \mathbb{C}$, $i \geqslant 1$, is defined by the formula

$$\lambda = (1 - y_i)(1 - y_i y_{i+1})\cdots(1 - y_i \cdots y_{i+d-2} y_{i+d-1}).$$

Denote by G^i the composition of the self-map G: $G^1(\mathbf{y}) = G(\mathbf{y})$, $G^{i+1}(\mathbf{y}) = G(G^i(\mathbf{y}))$ for $i \geqslant 1$. Recall that a point $\mathbf{y}$ is called a periodic point of period k if $G^k(\mathbf{y}) = \mathbf{y}$ and $G^i(\mathbf{y}) \neq \mathbf{y}$ for every $1 \leqslant i < k$. A periodic point of period 1 is called a fixed point.

A fixed point of the self-map G is the point $\boldsymbol{\gamma} = (\gamma, \ldots, \gamma) \in \mathbb{C}^{d-1}$ where γ is the root of the equation

$$\lambda = (1 - \gamma)(1 - \gamma^2)\cdots(1 - \gamma^d) \tag{2}$$

which has $d(d+1)/2$ many roots counted with multiplicity. Each fixed point γ corresponds to the constant y-sequence $\gamma, \gamma, \ldots$, and to the "spiral" x-sequence

$$x_0, \gamma x_0, \gamma^2 x_0, \ldots. \tag{3}$$

Remark 1. In the case $\lambda = 1$ (Gauss–Seidel), Eq. (2) has an exceptional root $\gamma = 0$ which does not give a meaningful orbit neither in y-space nor in x-space. If $\lambda = 0$, the dynamics in the x-space is trivial: F is the identity map. But the y-dynamics is nontrivial: the fixed point generator γ is the kth root of unity, $\gamma^k = 1$, with $1 \leqslant k \leqslant d$.

Remark 2. By (2), λ can be considered as a function of γ. This means that there exists, for every $\gamma \in \mathbb{C}$, a unique relaxation parameter $\lambda \in \mathbb{C}$ such that the self-map G has a fixed point γ. If ζ is the kth root of unity, $1 \leqslant k \leqslant d/2$, then $\gamma = \zeta$ is a branch point of the function (2) with multiplicity $\lfloor d/k \rfloor$, where $\lfloor x \rfloor$ denotes the maximum integer that is not greater than x.

If $|\gamma| > 1$, the spiral x-sequence (3) tends to ∞. This is unsatisfactory as a polynomial root finder because it is expected that the sequence converges to the roots of the polynomial $p(x) = x^d$, which is the origin. We will see that such a diverging sequence is "unstable", and hence scarcely visible with an ordinary choice of initial values.

Definition 3. Let $\boldsymbol{y}$ be a fixed point of the self-map G. Denote by $J(\boldsymbol{y})$ the Jacobian matrix of G at $\boldsymbol{y}$. An eigenvalue of $J(\boldsymbol{y})$ is called a multiplier of $\boldsymbol{y}$. The fixed point $\boldsymbol{y}$ is called
- stable if absolute values of all the multipliers of $\boldsymbol{y}$ are less than 1;
- unstable if absolute values of all the multipliers of $\boldsymbol{y}$ are greater than 1;
- a saddle if some absolute value of multipliers are less than 1, and some are greater than 1;
- central if some multiplier of $\boldsymbol{y}$ is of absolute value 1.

We call the spiral sequence (3) a stable spiral if the corresponding fixed point $\boldsymbol{\gamma}$ in the y-space is stable. Similarly for an unstable, etc., spiral.

Recall that $\zeta \in \mathbb{C}$ is a kth primitive root of unity if $\zeta^k = 1$ and $\zeta^i \neq 1$ for every $1 \leqslant i < k$. For a positive integer d, let

$$\begin{aligned}\varphi(d) &= \#\{1 \leqslant n < d \mid n \text{ and } d \text{ are relatively prime}\} \\ &= \#\{\zeta \in \mathbb{C} \mid \zeta \text{ is a primitive } d\text{th root of unity}\}\end{aligned}$$

be Euler's function. We will denote by G_λ the self-map G with the relaxation parameter λ.

Theorem 4. *Let $d \geqslant 2$. For each primitive dth root of unity ζ, there exists a function $\lambda \mapsto \gamma = \gamma(\lambda) \in \mathbb{C}$ defined for small $\lambda > 0$ such that*

$$\gamma(\lambda) = \zeta - \frac{\zeta}{d^2}\lambda + \mathrm{O}(|\lambda|^2) \quad \text{as } \lambda \to 0, \tag{4}$$

and that the point $\gamma(\lambda)=(\gamma(\lambda),\dots,\gamma(\lambda))$ *is a stable fixed point of* G_λ *with the multipliers* t_k, $1\leqslant k\leqslant d-1$, *where*

$$t_k=\zeta^k-\frac{k\zeta^k}{d^2}\lambda+\mathrm{O}(|\lambda|^2)\quad \text{as } \lambda\to 0. \tag{5}$$

Conversely, for small $\lambda>0$, *a stable fixed point* γ *of the self-map* G_λ *is equal to* $\gamma(\lambda)$ *for some primitive dth root of unity* ζ.

Corollary 5. *Let* $d\geqslant 2$ *and* $\lambda>0$ *be small. Then we have*

$$\#\{\gamma\in\mathbb{C}\mid \gamma \text{ is a stable fixed point of } G_\lambda\}=\varphi(d). \tag{6}$$

If γ *is a stable fixed point of* G_λ, *the corresponding x-spiral* (3) *converges to the origin.*

Proof of Theorem 4. The Jacobian matrix of G_λ at the fixed point γ is written by

$$J(\gamma)=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & & & \cdots & 0\\ 0 & & & \cdots & 1\\ \phi_1 & \phi_2 & & \cdots & \phi_{d-1}\end{pmatrix},$$

where

$$\phi_i:=\left.\frac{\partial g}{\partial y_i}\right|_{\gamma}=-\frac{1-\gamma^d}{\gamma^d}\sum_{k=i}^{d}\frac{\gamma^k}{1-\gamma^k},\qquad 1\leqslant i\leqslant d-1.$$

The multipliers of γ are the roots of the polynomial

$$\begin{aligned}E_\gamma(t)&=t^{d-1}-\phi_{d-1}t^{d-2}-\cdots-\phi_1\\ &=\frac{1-\gamma^d}{\gamma^d}\sum_{i=0}^{d-1}t^i\sum_{j=i+1}^{d}\frac{\gamma^j}{1-\gamma^j}=\frac{1-\gamma^d}{\gamma^d}\sum_{j=1}^{d}\frac{\gamma^j}{1-\gamma^j}\sum_{i=0}^{j-1}t^i.\end{aligned}$$

If $\lambda>0$ is small, then, by (2), γ is close to a jth primitive root of unity ζ, where $1\leqslant j\leqslant d$. We claim that if $j\neq d$, $E_\gamma(t)$ has a root with absolute value greater than 1, showing that the fixed point γ is not stable. First, consider the case that $1\leqslant j<d$ and j divides d. Let $d=jq$. We have

$$\lim_{\gamma\to\zeta}E_\gamma(t)=q\sum_{k=1}^{q}\frac{1}{k}\sum_{i=0}^{jk-1}t^i,$$

which has the top term t^{d-1} and the constant term $q\sum_{k=1}^{q}\frac{1}{k}>1$. Thus $E_\gamma(t)$ has a root with absolute value greater than 1 if γ is close to ζ. Second, consider the case that j does not divide d. Let $d=qj+r$ with $0<r<j$. We have

$$\lim_{\gamma\to\zeta}(1-\gamma^j)\phi_1=-\frac{1-\gamma^r}{\gamma^r}\sum_{k=1}^{q}\frac{1}{k}\neq 0.$$

Thus, $\lim_{\gamma\to\zeta}|\phi_1| = +\infty$, and the roots of $E_\gamma(t)$ is not bounded when $\gamma \to \zeta$.

So let ζ be a primitive dth root of unity. In (2), it is easy to see that

$$\left.\frac{\mathrm{d}\lambda}{\mathrm{d}\gamma}\right|_{\gamma=\zeta} = -\frac{d^2}{\zeta} \neq 0. \tag{7}$$

Thus, the local inverse of the function (2) is well defined around $\gamma=\zeta$. We denote it by $\lambda \mapsto \gamma(\lambda)$. Then we see that (4) holds, and that $|\gamma(\lambda)|<1$ for small $\lambda>0$.

It remains to show that the multipliers of $\gamma(\lambda)$ are as in (5), because it then immediately follows that $|t_k|<1$ for small $\lambda>0$. Fix $1\leqslant k\leqslant d-1$ and let

$$\begin{aligned}\hat{E}(\gamma,t) := E_\gamma(t) &= \sum_{i=0}^{d-1} t^i + \frac{1-\gamma^d}{\gamma^d}\sum_{j=0}^{d-1}\frac{\gamma^j}{1-\gamma^j}\sum_{i=0}^{j-1} t^i \\ &= \prod_{i=1}^{d-1}(t-\zeta^i) + \frac{1-\gamma^d}{\gamma^d}\sum_{j=1}^{d-1}\frac{\gamma^j}{1-\gamma^j}\frac{1-t^j}{1-t}.\end{aligned}$$

We have $\hat{E}(\zeta,\zeta^k)=0$. The local function $\gamma \mapsto t=t(\gamma)$ such that $t(\zeta)=\zeta^k$ is well defined because

$$\left.\frac{\partial \hat{E}}{\partial t}\right|_{(\gamma,t)=(\zeta,\zeta^k)} = \prod_{i=1,i\neq k}^{d-1}(\zeta^k-\zeta^i) = \frac{d}{\zeta^k(\zeta^k-1)} \neq 0.$$

Moreover, we have

$$\begin{aligned}\left.\frac{\partial \hat{E}}{\partial \gamma}\right|_{(\gamma,t)=(\zeta,\zeta^k)} &= -\frac{d}{\zeta}\sum_{j=1}^{d-1}\frac{\zeta^j}{1-\zeta^j}\frac{1-\zeta^{kj}}{1-\zeta^k} \\ &= -\frac{d}{\zeta}\frac{1}{1-\zeta^k}\sum_{j=1}^{d-1}\sum_{i=1}^{k}\zeta^{ij} \\ &= -\frac{kd}{\zeta(1-\zeta^k)}.\end{aligned}$$

Hence,

$$\left.\frac{\mathrm{d}t}{\mathrm{d}\gamma}\right|_{\gamma=\zeta} = -\left.\frac{\partial \hat{E}}{\partial \gamma}\middle/\frac{\partial \hat{E}}{\partial t}\right|_{(\gamma,t)=(\zeta,\zeta^k)} = k\zeta^{k-1}$$

and

$$\left.\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right|_{\lambda=0} = \left.\frac{\mathrm{d}t}{\mathrm{d}\gamma}\middle/\frac{\mathrm{d}\lambda}{\mathrm{d}\gamma}\right|_{\gamma=\zeta} = -\frac{k\zeta^k}{d^2}. \qquad \square$$

3. Experiments

3.1. Number of stable fixed points

Experiments that show that Corollary 5 is still valid for $\lambda = 1$ (Gauss–Seidel).

Let $\lambda = 1$. Table 1 shows the number of fixed points γ, according to the corresponding degree d, norm $|\gamma|$, and the stability of the point $\gamma = (\gamma, \dots, \gamma)$ under the dynamics G. We see that (6) holds, and that there are no stable fixed points with $|\gamma| \geqslant 1$.

Figs. 1–3 present the contour-plots over the complex γ-plane bounded by $-2 < \Re(\gamma) < 2$, $-2 < \Im(\gamma) < 2$, which show the stability of the fixed point γ under the dynamics G_λ where λ is determined as a function of γ by (2). The unit circle with center at the origin is also plotted. The γ-plane is divided into three open sets (and their boundary): a bounded simply connected *black* domain consisting of $\varphi(d)$ connected components (where γ is stable), a bounded *white* domain which is possibly multiply-connected (where γ is a saddle), and an unbounded *gray* domain (where γ is unstable under G_λ). Each connected component of the *black* domain contains an inverse image of$\lambda = 1$ in its interior (marked $+$), and a primitive dth root of unity on its boundary. Fig. 1 is $d = 3$, Fig. 2 is $d = 6$, and Fig. 3 is $d = 7$.

Table 1
Cardinality of fixed spirals, Gauss–Seidel Durand–Kerner's method

	$\|\gamma\| < 1$			$\|\gamma\| > 1$		
Degree	Stable	Saddle	Unstable	Saddle	Unstable	Total
3	2	1	0	0	2	5
4	2	3	0	0	4	9
5	4	5	0	0	5	14
6	2	9	0	0	9	20
7	6	11	2	0	8	27
8	4	13	2	4	12	35
9	6	15	2	6	15	44
10	4	21	2	6	21	54
11	10	23	2	8	22	65
12	4	33	2	4	34	77
13	12	31	2	10	35	90
14	6	47	4	4	43	104
15	8	47	4	10	50	119
16	8	59	4	6	58	135
17	16	59	4	12	61	152
18	6	79	4	8	73	170
19	18	69	6	18	76	187
20	8	87	4	20	90	209
21	12	91	6	26	95	230
22	10	104	7	28	103	252
23	22	116	7	24	106	275
24	8	128	7	36	120	299
25	20	130	7	43	124	324

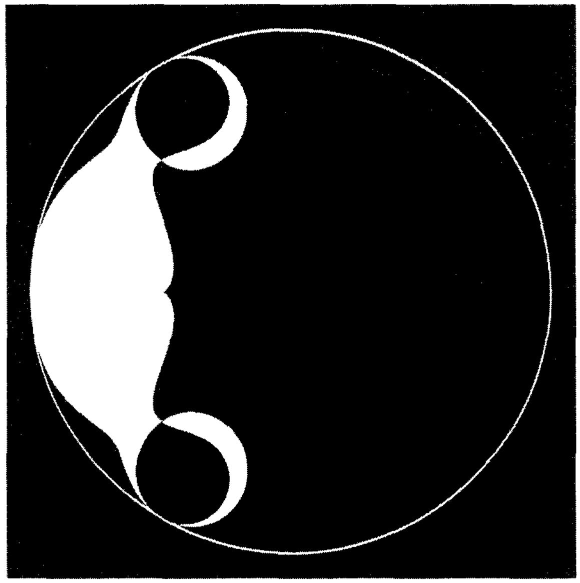

Fig. 1. Stability of the fixed spiral, γ-contour-plot. $d = 3$.

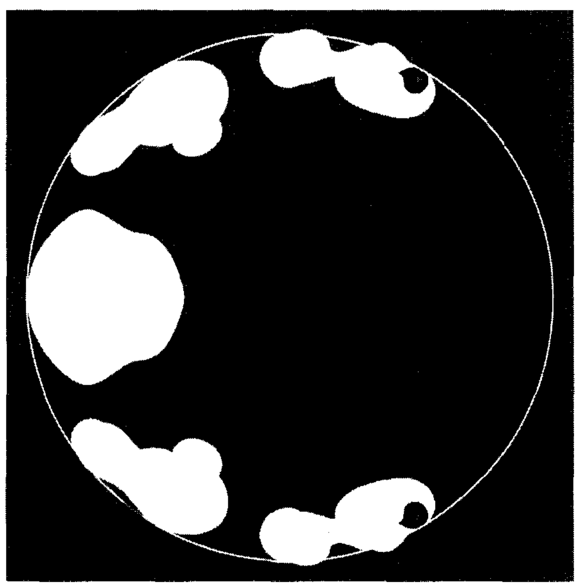

Fig. 2. Stability of the fixed spiral, γ-contour-plot. $d = 6$.

3.2. Phase space separatrix

Separatrices of dynamics G with $\lambda = 1$.
Figs. 4–6 present the complex one-dimensional subset of the y-space defined by

$$\{\boldsymbol{y} = (y, \dots, y) \in \mathbb{C}^{d-1} \mid -2 < \Re(y) < 2, -2 < \Im(y) < 2\}.$$

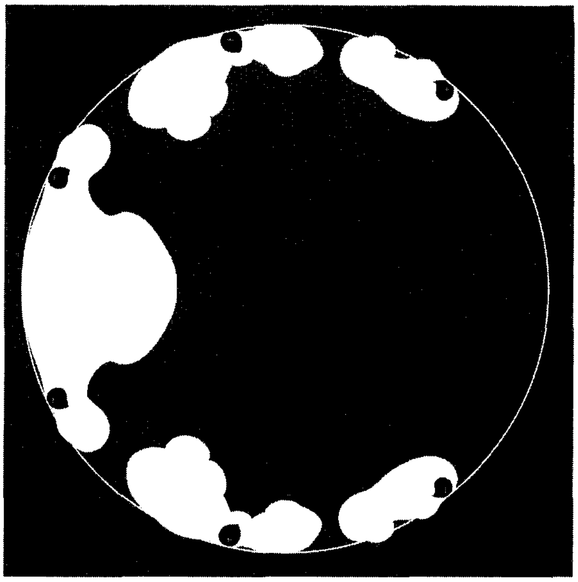

Fig. 3. Stability of the fixed spiral, γ-contour-plot. $d = 7$.

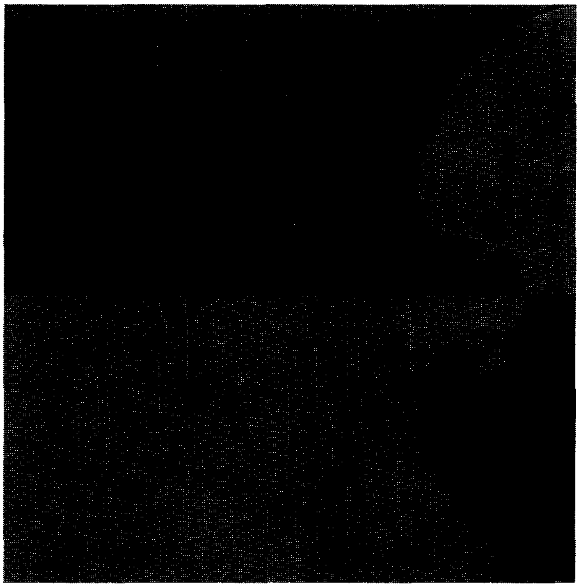

Fig. 4. Separatrix, $d = 3$.

The *gray* regions are the attracting regions of stable fixed orbits under the dynamics G. The *white* regions are the attracting regions of stable periodic orbits of period greater than 1. Every stable periodic orbit that we detected has period d. The mark $+$ shows the coordinate of γ that generates the stable fixed point γ.

Fig. 4 is $d = 3$. There are $\varphi(3) = 2$ regions with the common boundary. Fig. 5 is $d = 5$, $\varphi(5) = 4$. Fig. 6 is $d = 6$, $\varphi(6) = 2$.

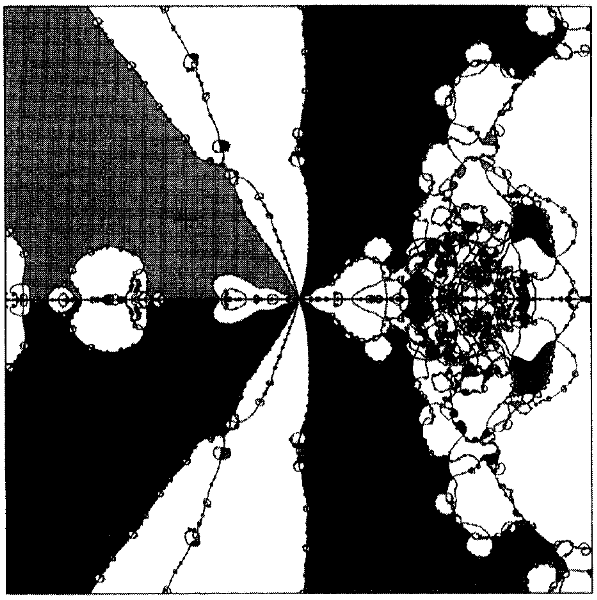

Fig. 5. Separatrix, $d=5$.

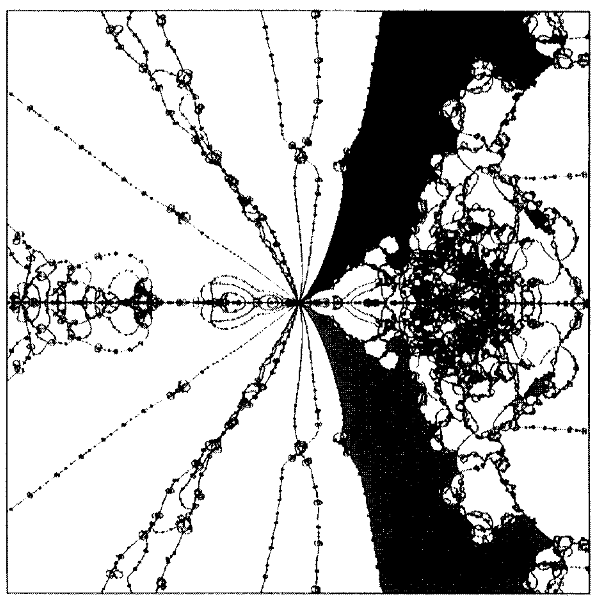

Fig. 6. Separatrix, $d=6$.

3.3. Stable spiral orbits

Figs. 7–10 are examples of orbits $\{x_i \in \mathbb{C} \mid 0 \leqslant i < 20\}$ corresponding to the stable periodic orbits of G, with $d=5$, $\lambda=1$, $x_0=1$. Fig. 7 corresponds to the stable fixed point of G studied in Section 2.

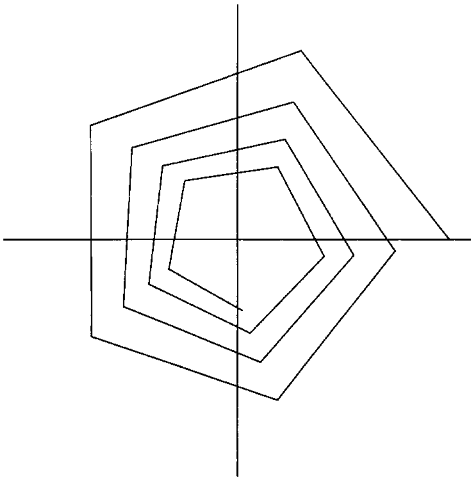

Fig. 7. Stable "fixed" spiral. $d=5$.

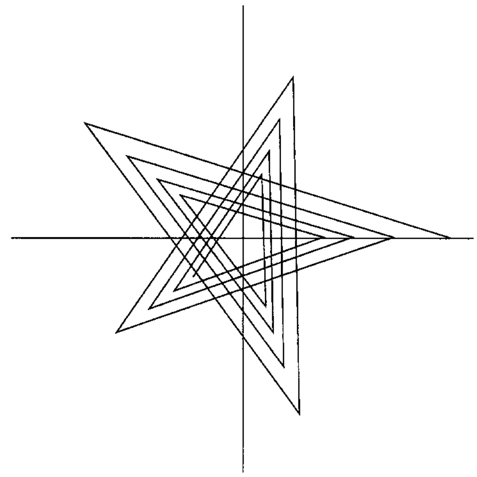

Fig. 8. Stable spiral of "period" 5. $d=5$.

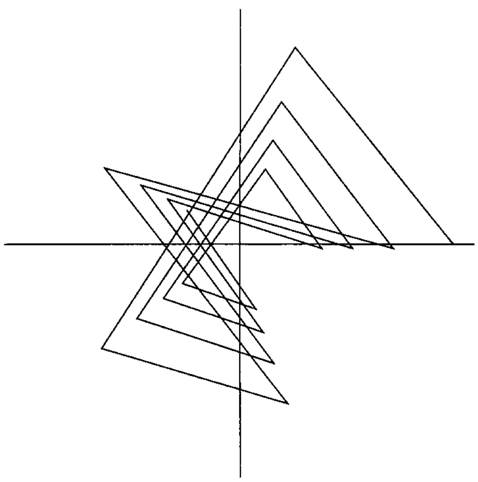

Fig. 9. Stable spiral of "period" 5. $d=5$.

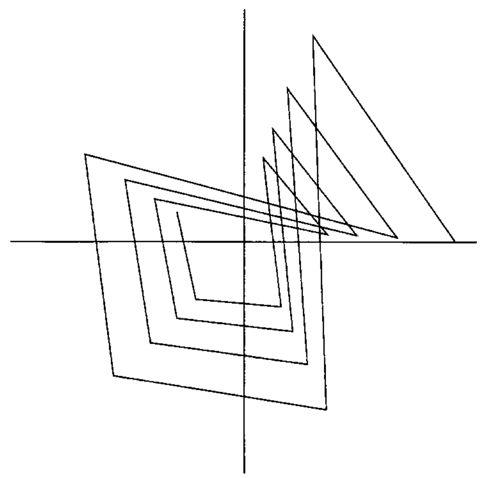

Fig. 10. Stable spiral of "period" 5. $d=5$.

Appendix

Durand–Kerner's method of its original version for the monic complex polynomial $p(z)$ of degree $d>1$ is the rational function $F : \mathbb{C}^d \to \mathbb{C}^d$ defined by

$$\pi_i F = x_i - \frac{p(x_i)}{\prod_{k=1,k\neq i}^{d}(x_i - x_k)},$$

where $\pi_i : \mathbb{C}^d \to \mathbb{C}$ is the projection defined by $\pi_i(x_1,\dots,x_d)=x_i$.

Miyakoda [5] showed that for each permutation σ of the finite set of indices $\{1,\dots,d\}$ and for each primitive dth root ζ of unity, the point $(\zeta^{\sigma(1)},\dots,\zeta^{\sigma(d)})$ is "stable" under the dynamics of the phase space of "relative configurations" of dimension $d-1$. The corresponding x-orbit has a regular d-gon configuration at each step of iteration, and it is called a "balanced convergence".

References

[1] K. Dochev, An alternative method of Newton for simultaneous calculation of all the roots of a given algebraic equation, Phys. Math. J. Bulgar. Acad. Sci. 5 (2) (1962) 136–139.

[2] I.E. Durand, Solutions Numérique des Équations Algébriques. Tome I: Équations du Type $F(x)=0$; Racines d'une Polynôme, Masson, Paris, 1960, pp. 279–281.

[3] S. Kanno, T. Yamamoto, Validated computation of polynomial zeros by the Durand–Kerner method, in: J. Herzberger (Ed.), Topics in Validated Computations, Elsevier, Amsterdam, 1994, pp. 55–63.

[4] I.O. Kerner, Ein Gesamtschrittverfahren zur Berechnung der Nullstellen von Polynomen, Numer. Math. 8 (1966) 290–294.

[5] T. Miyakoda, Balanced convergence of iterative methods to a multiple zero of a complex polynomial, J. Comput. Appl. Math. 39 (1992) 201–212.

[6] M.D. Prešić, Un procédé itératif pour déterminer k zéros d'un polynôme, Comptes Rendus de l'Academie des Science, Paris, Ser. I 273 (1971) 446–449.

[7] K. Weierstrass, Neuer Beweis des Satzes, dass jede ganze rationale Function einer Veränderlichen dargestellt werden kann als ein Product aus lineare Functionen derselben Veränderlichen, Werke, Johnson Reprint Corp., New York, 1967, pp. 251–269.